Problem 1 (*45 minutes; 15 points in total*)

Answer the following questions, brief and to the point:

2 pnts (a) Evaluate the commutator $[L_i, p_j]$.

2 pnts (b) Calculate the wavelength in nm of the Balmer H-β line ($n = 4 \rightarrow n = 2$) in hydrogen (use $\hbar c = 197$ eV nm). What is the color of this line?

2 pnts (c) Which two physical effects are responsible for the fine-structure of the hydrogen spectrum?

2 pnts (d) Explain the principle of a nuclear magnetic resonance (NMR) experiment and how it can be used to measure the g-factor of the proton.

2 pnts (e) Which of the two isotopes of rubidium ($Z = 37$), ^{86}Rb or ^{87}Rb, can be used for Bose-Einstein condensation? Why?

2 pnts (f) Show that the ground-state electronic configuration $^7S_3$ of Cr ($Z = 24$, [Ar] $3d^5\,4s^1$) does not violate the Pauli principle.

2 pnts (g) What are para- and orthohelium? Sketch the energy levels (no fine-structure) of both. Explain the differences.

1 pnt (h) Calculate the Bohr radius of hydrogen, $a_0 = \hbar^2/(me^2)$, in nm.

137
137 ×
959
4110
13700
18769
16 ×
112614
187690
300304

Problem 2 (*45 minutes; 15 points in total*)

At a particular time, the wave function of a spin-1/2 particle moving in a three-dimensional potential is

$$\psi(\vec{r}) = A(x+y+z)\, e^{-\beta|\vec{r}|}\, \xi\ , \quad \text{where } \xi = \frac{1}{\sqrt{5}}\begin{pmatrix} i \\ -2 \end{pmatrix}$$

is its spinor, with respect to the basis α, β of eigenvectors of S_z.

3 pnts (a) Write the spatial part of ψ in terms of spherical harmonics Y_ℓ^m, by using spherical coordinates for $\vec{r} = (x, y, z)$.

2 pnts (b) Calculate the value of the normalization constant A (assume that A is real and positive).

3 pnts (c) If we were to measure L^2 and L_z, what values could we find, and with what probability? What is the expectation value of L_z?

2 pnts (d) If we measure for L_z a value of 0, what will the new wave function be?

3 pnts (e) If we were to measure S_z, what values could we find, and with what probability? The same question for S_x; first give its eigenvectors α_x, β_x on the basis α, β.

2 pnts (f) $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum of the particle. If we were to measure J^2, what values could we find?

To solve problem (a), you should use that

$$\begin{aligned} Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta\ , \\ Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}\ . \end{aligned}$$

For problem (b), you may use that

$$\int_0^\infty r^n\, e^{-\beta r} dr = n!\, \beta^{-(n+1)}\ .$$

Problem 3 (*35 minutes; 10 points in total*)

An electron, with mass m, is confined in a 3D cubic box with sides of length L, *i.e.* the potential is:

$$
\begin{aligned}
V(x,y,z) &= 0 \qquad 0 < x,y,z < L \ , \\
&= \infty \qquad x,y,z < 0 \ \text{or} \ x,y,z > L \ .
\end{aligned}
$$

3 pnts (a) Give the (time-independent) Schrödinger equation. Show that the solution that obeys the proper boundary conditions is

$$\psi(x,y,z) = A\sin(k_x x)\sin(k_y y)\sin(k_z z) \ .$$

What are the conditions on k_x, k_y, and k_z? Give the corresponding energy eigenvalues E. Calculate the normalization constant A (assume that it is real and positive).

2 pnts (b) Discuss the degeneracy of the energy levels.

3 pnts (c) Now put 24 electrons in the box. Assume that they do not interact with each other. What is the lowest possible energy, in units of $\hbar^2\pi^2/(2mL^2)$?

2 pnts (d) Answer question (c) for *spinless* particles with mass m.

$$\sin^2 x = \left(\frac{1}{2i}\right)^2 (e^{ix} - e^{-ix})(e^{ix} - e^{-ix})$$

$$= \frac{-1}{4}\left(e^{2ix} + e^{-2ix} - 2\right)$$

$$= -\frac{\cos 2x}{2} + \frac{1}{2}.$$

$$=$$

Problem 4 (*35 minutes; 10 points in total*)

An electron is at rest at the origin in the presence of a magnetic field whose magnitude (B_0) is constant but whose direction rotates around in the (x, y) plane at constant angular velocity α, so

$$\vec{B}(t) = B_0 \left[\cos(\alpha t)\hat{x} + \sin(\alpha t)\hat{y}\right] \ . \tag{1}$$

The Hamiltonian for the particle is given by $H = (e/m)\vec{B} \cdot \vec{S}$, where $\vec{S} = \hbar\vec{\sigma}/2$ are the spin matrices. A possible solution is given by the spinor

$$\xi(t) = \begin{pmatrix} [\cos(\lambda t/2) + i(\alpha/\lambda)\sin(\lambda t/2)]e^{-i\alpha t/2} \\ i(\omega/\lambda)\sin(\lambda t/2)e^{i\alpha t/2} \end{pmatrix} \tag{2}$$

where $\omega = -eB_0/m$ and $\lambda = \sqrt{\alpha^2 + \omega^2}$.

2 pnts (a) Write the Hamiltonian explicitly as a 2×2 matrix.

3 pnts (b) Show that $\xi(t)$ is indeed a solution of the time-dependent Schrödinger equation for this problem.

1 pnt (c) Verify that $\xi(t)$ is properly normalized.

2 pnts (d) Calculate $\langle\sigma_z\rangle$ to verify that $\xi(t = 0)$ corresponds to a spin-up electron.

2 pnts (e) Calculate the expectation value of the spin in the y-direction as a function of time.